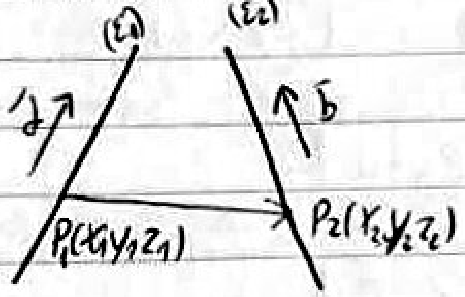
Π: 6/4/'17

- Σχετικές θέσεις 2 ευθειών στο χώρο -

Δίνονται οι $(\varepsilon_1), (\varepsilon_2)$ οι οποίες διέρχονται από τα σημεία $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ και είναι παράλληλες προς τα δ/τα $\bar{\alpha} = (\alpha_1, \beta_1, \gamma_1)$, $\bar{\beta} = (\alpha_2, \beta_2, \gamma_2)$ αντίστοιχα.



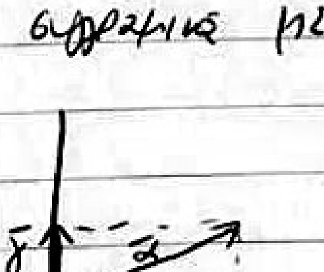
$$(\varepsilon_1) \quad \frac{x-x_1}{\alpha_1} = \frac{y-y_1}{\beta_1} = \frac{z-z_1}{\gamma_1}$$

$$(\varepsilon_2) \quad \frac{x-x_2}{\alpha_2} = \frac{y-y_2}{\beta_2} = \frac{z-z_2}{\gamma_2}$$

- $\bar{\alpha}, \bar{\beta}$ γ.ε. $\Leftrightarrow (\varepsilon_1) \equiv (\varepsilon_2)$ ή $(\varepsilon_1) \parallel (\varepsilon_2)$

- $\bar{\alpha}, \bar{\beta}$ γ.α. $\Leftrightarrow (\varepsilon_1), (\varepsilon_2)$ τέμνονται ή είναι ασύμβατες.
  $(\varepsilon_1), (\varepsilon_2)$ είναι <u>συνεπίπεδες</u> $\Leftrightarrow \overline{P_1P_2}, \bar{\alpha}, \bar{\beta}$ συνεπίπεδα
  $\Leftrightarrow [\overline{P_1P_2}, \bar{\alpha}, \bar{\beta}] = 0$

Άσκηση Να αναλυθεί το δ/μα $\bar{\alpha} = (1, -5, 3)$ σε δύο κάθετες μεταξύ τους συνιστώσες εκ των οποίων είναι συγγραμμική με το $\bar{\beta} = (-2, 3, 1)$



Θέλω να βρω $\lambda \in \mathbb{R}$ και δ/μα $\bar{\gamma}$ τ.ω. $\bar{\alpha} = \bar{\gamma} + \lambda\bar{\beta}$ και $\bar{\gamma} \perp \bar{\beta}$

$\bar{\gamma} = \bar{\alpha} - \lambda\bar{\beta} \qquad \bar{\gamma} \perp \bar{\beta} \Leftrightarrow \langle \bar{\gamma}, \bar{\beta} \rangle = 0 \Leftrightarrow$

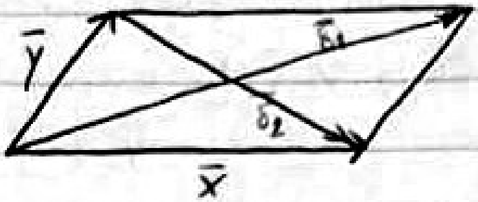
$\Leftrightarrow \langle \bar{\alpha} - \lambda\bar{\beta}, \bar{\beta} \rangle = 0 \Leftrightarrow \langle \bar{\alpha}, \bar{\beta} \rangle - \lambda \langle \bar{\beta}, \bar{\beta} \rangle = 0$

$$\Leftrightarrow \lambda = \frac{\langle \bar{\alpha}, \bar{\beta} \rangle}{\langle \bar{\beta}, \bar{\beta} \rangle} \Leftrightarrow \lambda = \frac{-2-15+3}{14} \Rightarrow \lambda = -1$$

άρα $\bar{\gamma} = \bar{\alpha} - \bar{\beta}$

Άσκηση Θεωρούμε τα μοναδ. διάν. $\bar{\alpha}, \bar{\beta}$ ώστε $(\widehat{\bar{\alpha}, \bar{\beta}}) = \frac{\pi}{4}$. Να βρεθεί το εμβ. του # με διαγωνίους $\bar{\delta}_1 = 7\bar{\alpha} - 3\bar{\beta}$, $\bar{\delta}_2 = -2\bar{\alpha} + \bar{\beta}$

Λύση



Το ζητούμενο εμβ. είναι
$E = |\bar{x} \times \bar{y}|$

$$\begin{cases} \bar{x} + \bar{y} = \bar{\delta}_1 \\ \bar{x} - \bar{y} = \bar{\delta}_2 \end{cases} \Leftrightarrow \begin{cases} 2\bar{x} = \bar{\delta}_1 + \bar{\delta}_2 \\ 2y = \bar{\delta}_1 - \bar{\delta}_2 \end{cases} \Leftrightarrow$$

$$\begin{cases} \bar{x} = \frac{1}{2}(\bar{\delta}_1 + \bar{\delta}_2) = \frac{1}{2}(7\bar{\alpha} - 3\bar{\beta} - 2\bar{\alpha} + \bar{\beta}) \\ \bar{y} = \frac{1}{2}(\bar{\delta}_1 - \bar{\delta}_2) = \frac{1}{2}(7\bar{\alpha} - 3\bar{\beta} + 2\bar{\alpha} - \bar{\beta}) \end{cases}$$

$$\bar{x} = \frac{1}{2}(5\bar{\alpha} - 2\bar{\beta}) \quad , \quad \bar{y} = \frac{1}{2}(9\bar{\alpha} - 4\bar{\beta})$$

$$\bar{x} \times \bar{y} = \frac{1}{4}(5\bar{\alpha} - 2\bar{\beta}) \times (9\bar{\alpha} - 4\bar{\beta})$$

$$= \frac{1}{4}\left(5\bar{\alpha} \times 9\bar{\alpha} - 5\bar{\alpha} \times 4\bar{\beta} - 2\bar{\beta} \times 9\bar{\alpha} + 2\bar{\beta} \times 4\bar{\beta}\right)$$

$$= 1/4\,(-20\,\alpha \times \bar{\beta} + 18\,\alpha \times \bar{\beta})$$

$$\boxed{\bar{x} \times \bar{y} = -\frac{1}{2}\,\alpha \times \bar{\beta}}$$

$$E = |\bar{x} \times \bar{y}| = 1/2\,|\bar{\alpha} \times \bar{\beta}| = 1/2\,|\bar{\alpha}||\bar{\beta}| \sin(\widehat{\bar{\alpha}, \bar{\beta}}) = 1/2 \cdot \sqrt{2}/2 = \sqrt{2}/4$$

Ασκηση Δινονται οι αριθμοί λ, μ με $|\lambda| \leq 1$, $|\mu| \leq 1$

Ν.α.ο. $\rightarrow$ $-1 \leq \lambda \cdot \sqrt{1-\mu^2} + \mu\sqrt{1-\lambda^2} \leq 1$.

Λύση

$|<\bar{\alpha}, \bar{\beta}>| \leq |\bar{\alpha}| |\bar{\beta}| \Leftrightarrow -|\bar{\alpha}||\bar{\beta}| \leq <\bar{\alpha}, \bar{\beta}> \leq |\bar{\alpha}|, |\bar{\beta}|$

Θεωρώ τα δ/τα

$\bar{\alpha} = (\lambda, \sqrt{1-\lambda^2})$, $\bar{\beta} = (\sqrt{1-\mu^2}, \mu)$ $, |\bar{\alpha}| = 1 = |\bar{\beta}|$

$<\bar{\alpha}, \bar{\beta}> = \lambda\sqrt{1-\mu^2} + \mu\sqrt{1-\lambda^2}$

Ασκηση Δινεται μοναδιαίο δ/μα $\bar{\alpha}$. Να λυθεί:

$<\bar{x}, \bar{\alpha}> = 1$ με άγνωστο το δ/μα $\bar{x}$ με $|\bar{x}| = 1$

Λύση

$<\bar{x}, \bar{\alpha}> = 1 \Leftrightarrow |\bar{x}||\bar{\alpha}| \cos(\bar{x}, \bar{\alpha}) = 1 \Leftrightarrow \cos(\bar{x}, \bar{\alpha}) = 1 \Leftrightarrow$

$(\bar{x}, \bar{\alpha}) = 0 \Leftrightarrow \bar{x} = \lambda\bar{\alpha}$, $\lambda > 0$.

$|\bar{x}| = |\lambda\bar{\alpha}| \Leftrightarrow 1 = \lambda \Leftrightarrow \lambda = 1$, η λύση είναι $\underline{\bar{x} = \bar{\alpha}}$

Ασκηση Να υπολογισ/ούν:

- $((\bar{\alpha} \times \bar{\beta}) \times \bar{\gamma}) \times \bar{\alpha}$
- $(\bar{\alpha} \times \bar{\beta}) \times (\bar{\gamma} \times \bar{\alpha})$
- $\bar{\alpha} \times (\beta \times (\bar{\gamma} \times \bar{\alpha}))$
- $<\bar{\alpha} \times (2\bar{\beta}), \bar{\alpha}>$

Λύση

i) $((\bar{\alpha} \times \bar{\beta}) \times \bar{\gamma}) \times \bar{\alpha} = (<\bar{\alpha}, \bar{\gamma}> \bar{\beta} - <\bar{\beta}, \bar{\gamma}> \bar{\alpha}) \times \bar{\alpha} =$

$= <\bar{\alpha}, \bar{\gamma}> \bar{\beta} \times \alpha - <\bar{\beta}, \bar{\gamma}> \cancel{\bar{\alpha} \times \bar{\alpha}} = <\bar{\alpha}, \bar{\gamma}> \bar{\beta} \times \bar{\alpha}$

$(\bar{\alpha} \times \bar{\beta}) \times (\bar{\gamma} \times \bar{\alpha}) = <\bar{\alpha}, \bar{\gamma} \times \bar{\alpha}> \bar{\beta} - <\bar{\beta}, \bar{\gamma} \times \bar{\alpha}> \bar{\alpha} =$

$-[\bar{\beta}, \bar{\gamma}, \bar{\alpha}] \bar{\alpha}$

$\bar{\alpha} \times (\bar{\beta} \times (\bar{\gamma} \times \bar{\alpha})) = \bar{\alpha} \times (<\bar{\beta}, \bar{\alpha}> \bar{\gamma} - <\bar{\beta}, \bar{\gamma}> \bar{\alpha}) =$

$= <\bar{\beta}, \bar{\alpha}> \bar{\alpha} \times \bar{\gamma} - <\bar{\beta}, \bar{\gamma}> \bar{\alpha} \times \bar{\alpha}$

$= <\bar{\beta}, \bar{\alpha}> \bar{\alpha} \times \bar{\gamma}$

<u>Άσκηση</u> Δίνονται τα δ/τα $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{b}_1, \bar{b}_2, \bar{b}_3$

ΝΔο

(i)

$$[\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3] \cdot [\bar{b}_1, \bar{b}_2, \bar{b}_3] = \begin{vmatrix} \langle \bar{\alpha}_1, \bar{b}_1 \rangle & \langle \bar{\alpha}_1, \bar{b}_2 \rangle & \langle \bar{\alpha}_1, \bar{b}_3 \rangle \\ \langle \bar{\alpha}_2, \bar{b}_1 \rangle & \langle \bar{\alpha}_2, \bar{b}_2 \rangle & \langle \bar{\alpha}_2, \bar{b}_3 \rangle \\ \langle \bar{\alpha}_3, \bar{b}_1 \rangle & \langle \bar{\alpha}_3, \bar{b}_2 \rangle & \langle \bar{\alpha}_3, \bar{b}_3 \rangle \end{vmatrix}$$

(ii) Επιπλέον δείξτε ότι:

$$[\bar{\alpha}, \bar{b}, \bar{\gamma}]^2 = \begin{vmatrix} |\bar{\alpha}|^2 & \langle \bar{\alpha}, \bar{b} \rangle & \langle \bar{\alpha}, \bar{\gamma} \rangle \\ \langle \bar{b}, \bar{\alpha} \rangle & |\bar{b}|^2 & \langle \bar{b}, \bar{\gamma} \rangle \\ \langle \bar{\gamma}, \bar{\alpha} \rangle & \langle \bar{\gamma}, \bar{b} \rangle & |\bar{\gamma}|^2 \end{vmatrix}$$

<u>Λύση</u>

i) Έστω $\bar{\alpha}_1 = (\alpha_{11}, \alpha_{12}, \alpha_{13})$, $\bar{\alpha}_2 = (\alpha_{21}, \alpha_{22}, \alpha_{23})$, $\bar{\alpha}_3 = (\alpha_{31}, \alpha_{32}, \alpha_{33})$

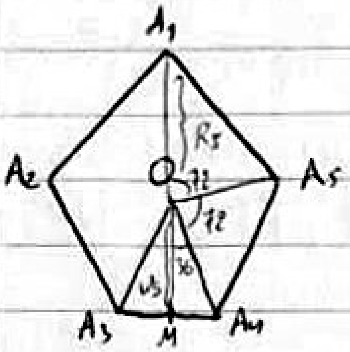
$\bar{b}_1 = (b_{11}, b_{12}, b_{13})$, $\bar{b}_2 = (b_{21}, b_{22}, b_{23})$, $\bar{b}_3 = (b_{31}, b_{32}, b_{33})$

$$[\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3][\bar{b}_1, \bar{b}_2, \bar{b}_3] = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} \begin{vmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{vmatrix} = \begin{vmatrix} \alpha_{11} b_{11} + \alpha_{12} b_{12} + \alpha_{13} b_{13} \end{vmatrix}$$

Άσκηση Δίνεται κανονικό πεντάγωνο $A_1A_2\ldots A_5$ κέντρου O.

ΝΔο $\rightarrow$ $\sum_{i=1}^{5} \overline{OA_i} = \overline{0}$



Λύση $\overline{OA_3} + \overline{OA_4} = 2\overline{OM}$, όπου M μέσο A_3A_4.

$A_1\hat{O}A_5 + A_5\hat{O}A_4 + A_4\hat{O}M = 180° \Rightarrow$

$\boxed{\underset{\lambda<0}{\overline{OM} = \lambda\overline{OA_1}}} \Rightarrow |\overline{OM}| = -\lambda|\overline{OA_1}| \Leftrightarrow υ_5 = -\lambda R_5$

$\lambda = -\frac{υ_5}{R_5}$ Δηλ. $\overline{OM} = -\frac{υ_5}{R_5}\overline{OA_1}$

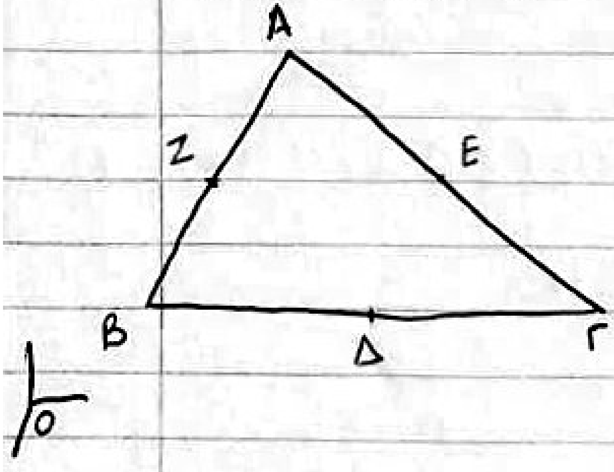
Άρα $\overline{OA_3} + \overline{OA_4} = 2\lambda\overline{OA_1}$ με $\lambda = -\frac{υ_5}{R_5}$

$$\left.\begin{aligned}
\overline{OA_1} + \overline{OA_2} &= 2\lambda\overline{OA_4}\\
\overline{OA_2} + \overline{OA_3} &= 2\lambda\overline{OA_5}\\
\overline{OA_3} + \overline{OA_4} &= 2\lambda\overline{OA_1}\\
\overline{OA_4} + \overline{OA_5} &= 2\lambda\overline{OA_2}\\
\overline{OA_5} + \overline{OA_1} &= 2\lambda\overline{OA_3}
\end{aligned}\right\} \overset{(+)}{\Longrightarrow} \quad 2(\overline{OA_1} + \overline{OA_2} + \ldots + \overline{OA_5}) = 2\lambda(\overline{OA_1} + \ldots + \overline{OA_5}) =$$

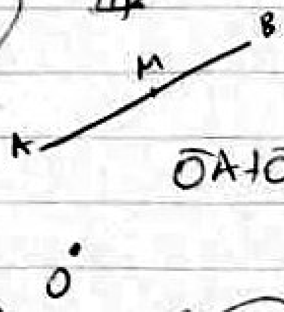
$$= 2\sum_{i=1}^{5} \overline{OA_i} - 2\lambda\sum_i \overline{OA_i} = \overline{0}$$

$$\Rightarrow \left(2 + 2\frac{υ_5}{R_5}\right)\sum_{i=1}^{5} \overline{OA_i} = \overline{0}$$

Άσκηση Δίνονται τα μέσα $Δ(4,4,0)$, $E(0,2,0)$, $Z(2,0,0)$ του $\overset{\Delta}{ABΓ}$. Να βρεθούν οι κορυφές του.



Tip:



$\overline{OA} + \overline{OB} = 2\overline{OM}$

(1) $\bar{OA}+\bar{OB} = 2\bar{OZ} = (4,0,0)$
(2) $\bar{OB}+\bar{O\Gamma} = 2\bar{O\Delta} = (8,8,0)$
(3) $\bar{O\Gamma}+\bar{OA} = 2\bar{OE} = (0,4,0)$

$$\xrightarrow{(+)} 2\bar{OA}+2\bar{OB}+2\bar{O\Gamma} = (12,12,0) \quad \boxed{\bar{OA}+\bar{OB}+\bar{O\Gamma} = (6,6,0)}\ (*)$$

$$(*) \xrightarrow{(1)} (4,0,0)+\bar{O\Gamma} = (6,6,0) \Leftrightarrow \bar{O\Gamma} = (2,6,0)$$

Άρα $\boxed{\Gamma(2,6,0)}$

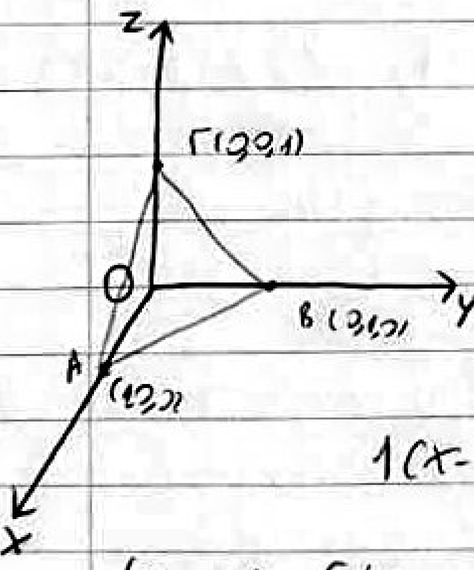
Ομοίως $(*) \xrightarrow{(2)} \dots$, $(*) \xrightarrow{(3)}$

Άσκηση Δίνονται τα $A(1,0,0)$, $B(0,1,0)$, $\Gamma(0,0,1)$
Βρείτε την i) εξ. επιπέδου που ορίζουν τα A,B,Γ.
Βρείτε ακόμη τις ii) παραμετρικές και iii) διανυσματικής εξίσωση

Λύση

Ελέγχω για γ.α ή γ.ε. $[\bar{OA},\bar{OB},\bar{O\Gamma}] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$

$\Rightarrow$ Γ.Α. $\Rightarrow A,B,\Gamma$ όχι συνευθειακά.



Το διάνυσμα $\bar{AB}\times\bar{A\Gamma} \perp$ στο (π)

$$\bar{AB}\times\bar{A\Gamma} = \begin{vmatrix} \bar{e_1} & \bar{e_2} & \bar{e_3} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \underset{\perp (\pi)}{(1,1,1)} \neq \bar{0}$$

Η αναλυτική εξίσωση του (π) του $A\hat{B}\Gamma$ είναι:

$$1(x-1)+1(y-0)+1(z-0)=0 \Leftrightarrow \boxed{x+y+z=1}$$

i) Αναλυτική εξ.

(x,y,z) διάνυσμα θέσης τυχαίου σημείου του (π).

$$(x,y,z) = (x,y,1-x-y) = (0,0,1)+(x,0,-x)+(0,y,-y)$$

$$(x,y,z) = (0,0,1) + \overset{\lambda}{x}(1,0,-1) + \overset{\mu}{y}(0,1,-1)$$

ii) παραμ. εξ.
$$\begin{cases} x = 0+\lambda\cdot 1+\mu\cdot 0 \\ y = 0+\lambda\cdot 0+\mu\cdot 1 \\ z = 1-\lambda\cdot 1-\mu\cdot 1 \end{cases}$$

iii) διαν. εξίσ.
$$\bar{r} = \bar{O\Gamma} + \lambda\bar{\alpha} + \mu\bar{\beta}, \quad \bar{\alpha} = (1,0,-1) = \bar{\Gamma A}, \quad \bar{\beta} = (0,1,-1) = \bar{\Gamma B}$$